Multiple CLFs for Stabilization of Nonlinear Systems with Input Constraints

Michael J. McCourt, Siddhartha S. Mehta, Zhen Kan, and J. Willard Curtis

***Abstract*—This paper considers stabilizing a nonlinear system with input constraints using switched control. The first result is that, for a given control Lyapunov function (CLF), a stabilizable set of initial conditions can be found. The paper then considers switching between multiple CLFs to increase the space of stabilizing inputs which also increases the set of stabilizable initial conditions. An algorithm is presented that identifies a subset of the known CLFs that may be used at the current time to produce stabilizing control inputs. While there are existing results for switched system stabilization using CLFs, stabilization for systems with input constraints is an original contribution. Sum of squares (SOS) methods are discussed for generating multiple CLFs, and examples are provided.**

I. Introduction

For the problem of stabilizing nonlinear systems, approaches that make use of control Lyapunov functions (CLFs) are widely used [1], [2]. Constructive formulas for selecting a control input using a CLF have been referred to as "universal formulas" [3] and have been applied to classes of systems with inputs constraints [4]–[6]. A related graphical interpretation of the CLF approach presented in [7] will be used in the current paper.

While many nonlinear systems can be stabilized using a single control input, the use of switching control has several benefits. Several controllers may be designed using competing criteria, e.g. robustness, optimality, convergence rate, bandwidth, etc. and a supervisory control scheme can switch between controllers to balance these criteria. Stability of the resulting switched system may be ensured using a common Lyapunov function approach [8], [9] or a multiple Lyapunov function (MLF) approach [10], [11]. There has been related work using CLFs for switched systems stability with a common CLF [12], [13] as well as multiple CLFs [14].

Finding a CLF may not be straightforward. However, the well-established computational methods of finding Lyapunov functions, linear matrix inequalities (LMIs) [15] for linear systems or sum of squares (SOS) optimization [16], [17] for polynomial nonlinear systems, may be leveraged for finding CLFs. One such approach for finding CLFs by using SOS is provided in [18]. Some discussion is given in the current paper on an alternative method for finding multiple CLFs. The method used assumes that a single stabilizing input is known, and it provides a set of CLFs which may be used to generate stabilizing control inputs.

This paper studies the problem of controlling an unstable nonlinear system using multiple CLFs that are known. At each time, the control input is determined from a single CLF. Switching control is introduced when switching from a control input determined by one CLF to a control input determined by another CLF. It is important to note that stabilizing nonlinear systems using switching control is a special case of switched systems stabilization. As such, there is prior work on this general stabilization problem, see e.g. [19], [20]. However, the current paper studies the problem of generating sets of stabilizing inputs using switching control for systems with input constraints. This problem has not been previously studied, to the best of the authors' knowledge.

The main contribution of this paper is in proposing a switching algorithm that guarantees stability. Rather than select a specific CLF, the algorithm is designed to identify a set of stabilizing CLFs. This approach has a common goal with the field of supervisory control where maximally permissive algorithms allow any switch that does not destabilize. The proposed algorithm is maximally permissive in that, at each time instant, no CLFs that are known to produce stabilizing control inputs are excluded from the set of allowable CLFs. An additional result guarantees the existence of a stabilizing control input for each CLF, when the initial state is constrained to a given set. A final contribution of this paper is in providing a computational approach, using SOS, for generating multiple CLFs for a given system.

This paper is organized as follows. Section II provides background material on CLFs and a decomposition that gives a set of stabilizing control inputs. Section III discusses the use of CLFs for systems with input constraints. The switching problem is introduced in Section IV along with a motivating example. The section continues with a switching algorithm that is maximally permissive and guarantees stability. Section V provides computational methods of generating CLFs. An example is given in Section VI. Concluding remarks are provided in Section VII.

II. Background Material

This paper considers nonlinear systems of the form,

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

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This research is supported in part by a grant from the Air Force Research Laboratory (AFRL) Mathematical Modeling and Optimization Institute, the Air Force Office of Scientific Research (AFOSR-LRIR), and the USDA NIFA AFRI National Robotic Initiative (NRI). Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the funding agency.

978-1-4799-8697-2/15 $31.00 © 2015 IEEE
DOI 10.1109/SMC.2015.20

where $x \in \mathcal{X} \subset \mathbb{R}^n$ is the system state with initial condition $x(t_0) = x_0$, and $u \in \mathcal{U} \subset \mathbb{R}^m$ is the system input. It is assumed that f is Lipschitz continuous and that $f(0) = 0$ to ensure that the origin is an equilibrium.

The CLF approach is based on the existence of a positive definite function $V : \mathbb{R}^n \to \mathbb{R}^+$ that is continuously differentiable, where $\mathbb{R}^+$ denotes the positive real numbers. Specifically, it is assumed that V may be bounded below and above by class-$\mathcal{K}$ functions $\underline{\alpha}$ and $\overline{\alpha}$,

$$\underline{\alpha}(||x||) \leq V(x) \leq \overline{\alpha}(||x||). \tag{2}$$

For more detail on class-$\mathcal{K}$ functions, refer to [21]. The system is stabilizable if

$$\inf_{u \in \mathcal{U}} \left\{ \frac{\partial V}{\partial x} \left[f(x) + g(x)u \right] \right\} < 0, \tag{3}$$

for all $x \in \mathcal{X}$ such that $x \neq 0$. For each state x an input u is denoted $u = \phi(x)$. For clarity, $\frac{\partial V}{\partial x} \in \mathbb{R}^{1\times n}$. It will be useful to denote the time derivative of V, $\dot{V} = \frac{\partial V}{\partial x}\dot{x}$. The CLF approach assumes that the given system has the small control property which is given in the following definition.

Definition 1. *[3] A system of the form (1) satisfies the small control property if* $\forall \epsilon_u > 0$, $\exists \delta_u > 0$ *such that,* $\forall x \in \mathcal{X}$ *such that* $||x|| < \delta_u$, $\exists u \in \mathcal{U}$ *such that* $||u|| < \epsilon_u$ *that stabilizes the system, i.e. satisfies 3.*

When there are no input constraints, finding a stabilizing input is trivial once a CLF is known, see e.g. [3]. The approach used in the current paper instead finds a set of stabilizing inputs so that the actual control applied to the system may be chosen from this set using additional criteria.

For a fixed x_t at time t, a set of stabilizing control inputs can be determined directly from a given CLF and the system dynamics (1). This set forms a half-plane and can be found by decomposing the control vector u_t into elements parallel and orthogonal to the vector $g^T(x_t)\left(\frac{\partial V}{\partial x}\right)_t^T$ as in [7],

$$u_t = -\alpha_t g^T(x_t) \left(\frac{\partial V}{\partial x} \right)^T_{x=x_t} + \eta_t, \tag{4}$$

where $\alpha_t \in \mathbb{R}$ is a constant scaling factor, and $\eta_t \in \mathbb{R}^m$ is the orthogonal component which satisfies $\left(\frac{\partial V}{\partial x}\right)_t g(x_t)\eta_t = 0$. As a result, a stabilizing control input can be determined at time t solely as a function of α_t. With this decomposition, a minimum stabilizing α_t can be written

$$(\alpha_t)_{\min} = \frac{\left(\frac{\partial V}{\partial x}\right)_{x=x_t} f(x_t)}{\left|\left| \left(\frac{\partial V}{\partial x}\right)_{x=x_t} g(x_t) \right|\right|^2}. \tag{5}$$

The system is marginally stable ($\dot{V} = 0$) when $u_t = (u_t)_{\min}$, which is when $\alpha_t = (\alpha_t)_{\min}$ and $\eta_t = 0$. For $\alpha_t > (\alpha_t)_{\min}$, $\dot{V} < 0$ and the system is asymptotically stable. This decomposition provides a constructive approach for finding a set of stabilizing control inputs for each $x \in \mathcal{X}$. As an alternative to Sontag's universal formula [3], this requires additional computation as the input must be computed pointwise as the state evolves. However, the approach using the decomposition has an advantage as it provides a set of stabilizing control inputs rather than a single input. This decomposition can be visualized for two-dimensional control inputs in Fig. 1. For a fixed $x(t)$, the vectors u_{min} and η are shown.

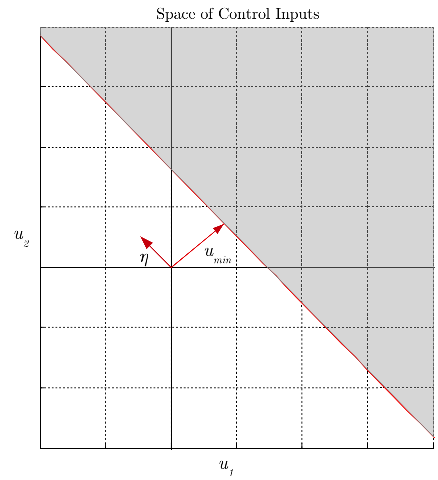

Fig. 1. The shaded half-plane represents a set of stabilizing control inputs that can be found directly from a control Lyapunov function and the given system dynamics by decomposition.

It should be noted that any approach that uses a CLF to generate control inputs is sufficient only for stability. The set of all stabilizing control inputs may be significantly larger. The motivation for the current paper comes from a desire to increase the known set of stabilizing control inputs by considering multiple CLFs.

III. CLFs for Systems with Input Saturation

As mentioned, there is existing work on constructive methods of finding a single control input using a CLF for classes of systems with input constraints [4]–[6]. A contribution of the current paper is in providing a set of stabilizing inputs using the given CLF. Specifically, this section uses the control-space decomposition covered previously to find a set of stabilizing inputs. The key problem in this direction is in guaranteeing that a region of inputs exists within the specified input bound. In order to show this result, the following assumption is made. The notation $\hat{v} \in \mathbb{R}^m$ will be used to denote an arbitrary vector in $\mathbb{R}^m$ with unit magnitude, i.e. $||\hat{v}||_2 = 1$.

Assumption 1. *The input space* $\mathcal{U}$ *is closed, bounded, and includes a neighborhood (of radius* $r > 0$*) of the zero vector in* $\mathbb{R}^m$.

This assumption guarantees that there exists a constant $r > 0$ that is a lower bound on the size of the available control in each direction, i.e. there exists $r \in \mathbb{R}$ such that, $\forall \hat{v} \in \mathbb{R}^m$, there exists $u \in \mathcal{U}$ such that $u \geq r\hat{v}$.

Theorem 1. *Consider an unstable nonlinear system of the form (1) with positive definite CLF,* $V(x)$*. If the input space* $\mathcal{U}$ *satisfies Assumption 1 with constant* $r_2 > 0$*, then there exists a region of* $\mathbb{R}^n$ *that is stabilizable. Furthermore, the set of states that are ultimately stabilizable is* $\mathcal{X}_s$*, where*

$$\mathcal{X}_s = \left\{ x \in \mathbb{R}^n \mid ||x|| \leq \delta \right\}, \tag{6}$$

and $\delta > 0$ is given in the following proof.

Proof. By Assumption 1, there exists a constant $r_2 > 0$ such that, $\forall \hat{v} \in \mathbb{R}^m$, there exists $u \in \mathcal{U}$ such that $u \geq r_2 \hat{v}$. Likewise, the small control property (Definition 1) guarantees there exists a constant $r_1 > 0$ such that

$$\frac{|\frac{\partial V}{\partial x} f(x)|}{||\frac{\partial V}{\partial x} g(x)||} \leq r_1, \tag{7}$$

for some set of states. The constant r_1 can be made arbitrarily small by the same property to satisfy $r_1 < r_2$. Define the set of states that are pointwise stabilizable as,

$$\mathcal{X}_{ps} = \left\{ x \in \mathbb{R}^n \mid \frac{|\frac{\partial V}{\partial x} f(x)|}{||\frac{\partial V}{\partial x} g(x)||} < r_1 \right\}. \tag{8}$$

It is important to note that $x_0 \in \mathcal{X}_{ps}$ does not guarantee that $x(t) \in \mathcal{X}_{ps}$ for all t. This will be guaranteed with an additional condition. Choose $r_x > 0$ as the largest radius of a ball, centered at the origin, that is completely contained within $\mathcal{X}_{ps}$. The constant $\delta > 0$ may be chosen according to $\delta = \underline{\alpha}^{-1}(\overline{\alpha}(r_x))$ to define $\mathcal{X}_s \subset \mathcal{X}_{ps}$. By the definition of $\mathcal{X}_{ps}$ and $\mathcal{X}_s$, if $x_0 \in \mathcal{X}_s$, then $V(x) \leq \overline{\alpha}(r_x)$. This guarantees that $||x(t)|| \leq r_x$ for all t which implies $x(t) \in \mathcal{X}_{ps}$ for all t. As $x(t) \in \mathcal{X}_{ps}$ for all t, there will always exist an input $u \in \mathcal{U}$ of the form

$$u = -\alpha g^T(x) \left(\frac{\partial V}{\partial x}\right)^T \tag{9}$$

with $\alpha > \alpha_{min}$ given by (5). As such an input $u \in \mathcal{U}$ can be found for all $x \in \mathcal{X}_s$, the system is stabilizable despite the input being constrained to $\mathcal{U}$. □

This result allows for a stabilizable set of states $\mathcal{X}_s$ to be determined directly from the known restrictions on the input set $\mathcal{U}$. Alternatively, the result provides a method of constructively identifying a set of stabilizing control inputs provided the state is in $\mathcal{X}_s$. The set may be determined from the knowledge of $\mathcal{U}$ and written as a function of x,

$$\mathcal{U}_s(x) = \{u \in \mathcal{U} | u = (1+\epsilon_1) u_{min}(x) + \epsilon_2 \eta, \epsilon_1 > 0\}, \tag{10}$$

where $\epsilon_1 > 0$ ensures asymptotic stability, and ϵ_1 and ϵ_2 are chosen to ensure that $u \in \mathcal{U}$.

IV. Switched Stabilization Using Multiple CLFs

The next problem studied in this paper is designing stabilizing control inputs using multiple CLFs. For a given system (1), it is assumed that the autonomous system ($u = 0$), is open-loop unstable. It is also assumed that there exist a set of control Lyapunov functions $\{V_1, V_2, ..., V_N\}$ and class-$\mathcal{K}$ functions that satisfy the following for all V_i,

$$\underline{\alpha}_i(||x||) \leq V_i(x) \leq \overline{\alpha}_i(||x||). \tag{11}$$

Theorem 1 can be used to identify stabilizable subsets of the state space, $\{\mathcal{X}_s^{(1)}, \mathcal{X}_s^{(2)}, ..., \mathcal{X}_s^{(N)}\}$ from the CLFs. As a function of x, stabilizing input sets $\{\mathcal{U}_1(x), \mathcal{U}_2(x), ..., \mathcal{U}_N(x)\}$ can be determined by (10). The complete set of known stabilizing control inputs that has been found for each state x can be denoted

$$\mathcal{U}_s(x) = \bigcup_{i=1}^{N} \mathcal{U}_i(x). \tag{12}$$

The system can be controlled by choosing a CLF i and a corresponding control input,

$$u(t) = \phi_i(x(t)) \in \mathcal{U}_i(x), \tag{13}$$

at each time t. While the system may be stabilized by choosing control inputs solely in $\mathcal{U}_i$, it may be desirable to switch to a control input from space $\mathcal{U}_j$ given by CLF V_j. When the control input is switched in this manner, the resulting feedback system becomes a switched system.

A switched system is a family of dynamics with a rule that determines switching between them. At each time, a single subsystem is active and the system dynamics are time invariant between switches. In the case of switching between CLFs, each CLF of the system creates one active subsystem which will also be referred to as a "mode" of the switched system. When switching control is applied to the system (1), the closed loop dynamics can be given by

$$\dot{x} = f_\sigma(x), \tag{14}$$

where $\sigma : \mathbb{R}^+ \to \Sigma$ and Σ is the set of subsystems, $\Sigma = \{1, 2, ..., N\}$. The function $\sigma(t)$ is piecewise constant. It is important to note that switching between CLFs to generate a switched control input can produce an unstable state trajectory despite each input being stable if it were applied over the infinite time horizon as the following example demonstrates.

Example 1. *Consider a linear system*

$$\dot{x} = \begin{bmatrix} 0.2 & 1 \\ 0.5 & 0.1 \end{bmatrix} x + \begin{bmatrix} 1 & 10 \\ 10 & 1 \end{bmatrix} u \tag{15}$$

with two CLFs of the form $V_i(x) = x^T P_i x$ where

$$P_1 = \begin{bmatrix} 289.1 & 0.104 \\ 0.104 & 29.76 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} 2.387 & 0.379 \\ 0.379 & 99.20 \end{bmatrix}. \tag{16}$$

Either CLF can be used to generate a stabilizing control input using the decomposition covered earlier. Instead, both CLFs will be used and the control input switched between them. Considering initial condition $x(t_0) = [0.1, \quad 1]^T$, the inputs are chosen to be near $(u_t)_{min}$ in the stabilizing space. While at each time t, the input is stabilizing according to one CLF, the switching directly causes instability which can be seen in Fig. 2.

Due to the possibility of destabilization, stability is the primary concern for the switched control algorithm. The algorithm given in this paper permits the use of any CLF that will not destabilize the system. Narrowing the set of allowable CLFs, and thus the set of switches allowed, is essentially a supervisory control problem. A desirable property of such an algorithm is to generate the largest known set of control inputs that are stabilizing. This approach is commonly referred to as "maximally permissive" in supervisory control of hybrid systems, see e.g. [22]. The active subsystem may be

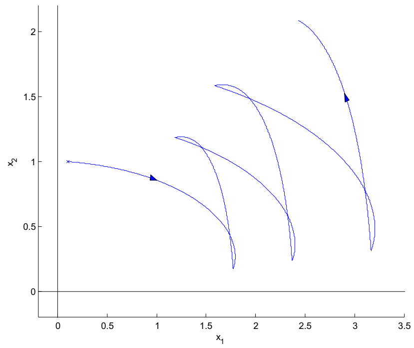

Fig. 2. The given trajectory begins at the point marked with an 'x'. The switching was chosen to generate an unstable trajectory by switching between two stabilizing controllers given by two different CLFs.

chosen from this set using additional considerations such as robustness, optimality, convergence rate, etc.

At a given time t the set of stabilizing control inputs $\mathcal{U}_i$ is determined by the CLF V_i. With the input restricted to $\mathcal{U}_i$, it can be shown that all $x \in \mathcal{X}_s^{(i)}$ are stabilizable using CLF V_i and Theorem 1. There are many situations when it is desirable to use control input that is not in $\mathcal{U}_i$ but is contained in $\mathcal{U}_j$ given by CLF V_j. The switch from synthesizing a controller using V_i to V_j may be allowed under certain conditions.

At each time, an allowable set of modes, denoted Σ_a, can be enumerated. The set $\Sigma_a \subset \Sigma$ is guaranteed not to be empty as the current mode i is stabilizing, i.e. $i \in \Sigma_a$. The algorithm provides a method of using all the known CLFs to construct the set Σ_a. It assumes that the initial state is restricted to be in the stabilizable set for at least one mode i, i.e.

$$x_0 \in \bigcup_{i=1}^{N} \mathcal{X}_s^{(i)}. \tag{17}$$

For each mode i, the algorithm tracks the value of the CLF $\left(V_{last}^{(i)}\right)$ at the last time the system switched to that mode. The algorithm continues while $t < t_f$, the final time. For analysis, we assume $t_f = \infty$ while in practice t_f will be finite.

Stability of the closed loop system (14) using the algorithm is shown using a MLF approach with the known set of CLFs. Theorem 1 is also used to guarantee the existence of a stabilizing control input within the constraints.

Theorem 2. *An unstable nonlinear system (1) may be stabilized using a switched control generated by a set of CLFs, if the active CLF V_i is always chosen according to Algorithm 1 and the control input from the set $\mathcal{U}_i$ determined by V_i .*

Proof. As (17) is satisfied, $\exists i$ such that $x_0 \in \mathcal{X}_s^{(i)}$. Using CLF V_i, Theorem 1 guarantees the existence of a stabilizing control input $u \in \mathcal{U}_i$. If no switching occurs, the system is stabilizable for this initial condition as, for each x, $\exists u \in \mathcal{U}_i$ that is stabilizing and ensures that $x(t) \in \mathcal{X}_s^{(i)}$ for all t.

A switch $i \to j$ may occur at time t if $x(t) \in \mathcal{X}_s^{(j)}$ and $V_j(x(t)) \le V_{last}^{(j)}$. The condition that $x(t) \in \mathcal{X}_s^{(j)}$ guarantees the existence of a stabilizing input after the switch according to Theorem 1. The condition $V_j(x(t)) \le V_{last}^{(j)}$ satisfies the Branicky non-increasing condition [10]. The algorithm guarantees that all subsequent switches follow these conditions, which guarantees that the system is asymptotically stable. □

Algorithm 1 Supervisory Control Algorithm

Initialize:
  Set $V_{last}^{(k)} = \infty, \forall k$
  Identify Σ_a where $i \in \Sigma_a$ if $x_0 \in \mathcal{X}_s^{(i)}$
  Choose $i \in \Sigma_a$
  Choose $u \in \mathcal{U}_i(x_0)$ given by V_i
  Set $V_{last}^{(i)} = V_i(x(t_0))$
while $t < t_f$ **do**
  for $j = 1$ **to** $j = N$ **do**
    if $x(t) \in \mathcal{X}_s^{(j)}$ **then**
      if $V_j(x(t)) < V_{last}^{(j)}$ **then**
        Add j to Σ_a
      end if
    end if
  end for
  Choose stay in i or switch to $j \in \Sigma_a$
  if switching to j **then**
    Set $V_{last}^{(j)} = V_j(x(t))$
    Set $i = j$
  end if
  Choose $u \in \mathcal{U}_i(x(t))$ by V_i
end while

Remark 1. *It is important to note that this approach is sufficient only for stability, as are all CLF approaches. While the union of the stabilizing input sets, $\mathcal{U}_s$, provides an enlarged set of known stabilizing inputs, it will not contain every stabilizing input, in general.*

Remark 2. *It is assumed that the CLFs $V_i(x)$ contain all known information about stabilizing control inputs. The only control input switching that is not allowed is switching that fails one of the assumptions of Theorem 2. As a result, the algorithm is maximally permissive in the sense that there are no known stabilizing inputs that are disallowed.*

Each iteration of the algorithm ends by providing a set of CLFs that may be used to generate the control input. The algorithm allows the flexibility that the actual CLF used may be chosen using additional considerations. One example is to choose the next mode based on which mode CLF has decreased by the largest percentage. This control policy will typically have a faster convergence rate than using a single CLF.

V. Computational Methods for Generating CLFs

The algorithm presented here is readily applicable to a given system assuming that a set of CLFs is known. However, it may be difficult in practice to find a CLF, let alone a set. For some classes of systems, there are methods to facilitate the search for CLFs. For linear systems there are methods of separating the search for a quadratic Lyapunov function and a stabilizing state feedback control [15]. For polynomial nonlinear systems, there are existing SOS methods to find

CLFs, see e.g. [18]. Instead of directly using these existing methods, we considered an alternative approach that assumes that one stabilizing state feedback control $u = \phi(x)$ is known. The input may be nonlinear but must be polynomial. The input may be parametrized by

$$\phi(x) = \sum_i k_i m_i(x) \tag{18}$$

where $k_i \in \mathbb{R}$ are constants and $\{m_i\}$ are all monomials that make up the function $\phi(x)$. This control may be applied to the system (1) to arrive at the closed loop dynamics given by,

$$\dot{x} = f_{cl}(x) = f(x) + g(x)\phi(x). \tag{19}$$

Using existing SOS methods, it is possible to determine a single CLF directly from the stabilized closed loop dynamics. The key to using SOS methods is to relax the search for positive definite or positive semi-definite functions to instead search for functions that can be written as a sum of squared terms. For example, the positive definite condition on the CLF, $V > 0, \forall x \neq 0$, can be relaxed to

$$V(x) = \sum_i a_i p_i(x)^2 \tag{20}$$

where $\{p_i(x)\}$ is a set of polynomials that is specified ahead of time. The optimization solver searches for the coefficients $a_i \geq 0$. It is clear that when a function can be written as the sum of squared terms, it must be positive semi-definite or positive definite. It should be noted that the relaxation from positive semi-definite to SOS is sufficient only.

As the search for a CLF is a feasibility problem, the cost to be minimized is arbitrary. The feasibility constraints are the system dynamics and restrictions on the Lyapunov function V. Specifically, these restrictions are the positive definite condition (20) and the condition that $\dot{V} \leq 0$, which is relaxed to the sufficient only SOS condition,

$$-\frac{\partial V}{\partial x}\left(f(x) + g(x)u\right) = \sum_j b_j p_j(x)^2. \tag{21}$$

This assumes that the system has polynomial dynamics, and a polynomial form must be chosen for the CLF. The semi-definite optimization solver searches for feasible coefficients for the CLF. The solver used to generate the CLFs in this paper is the SOSTOOL toolbox for MATLAB [17].

After a single CLF is found, the known control input can be perturbed by randomly varying the coefficients k_i in (18) which generates new closed loop dynamics. The same SOS methods can be repeated to find as many additional CLFs as desired by repeatedly perturbing the coefficients of the control input k_i.

VI. Example

This example demonstrates the approach presented in this paper. First, two CLFs are found using SOS methods. Then, sets of allowable initial conditions are determined and used in the stability preserving switching algorithm. Finally, the system is simulated with random switching, when allowed, to show that the resulting state trajectory is stable.

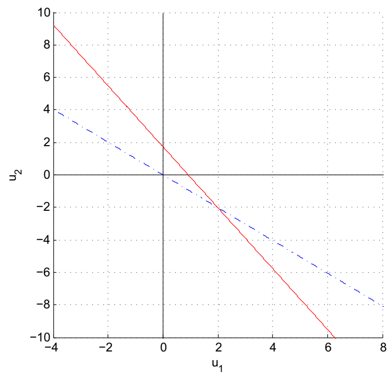

Fig. 3. The range of stabilizing inputs is demonstrated for two CLFs for a particular state $x = [-0.2, -0.2]^T$.

Example 2. *Consider a nonlinear system with dynamics given by (1) where*

$$f(x) = \begin{pmatrix} x_1^3 + x_2 \\ -x_1 + x_2^2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22}$$

It is known that one stabilizing input is

$$u(x) = \begin{pmatrix} -k_1 x_1^3 \\ -k_2 x_2 - x_2^2 \end{pmatrix}. \tag{23}$$

The input is stabilizing for $k_1 > 1$ and $k_2 > 0$. Using the controller, a Lyapunov function may be found for the closed loop system using the SOS method covered in the previous section. It is assumed that the CLF is of the form,

$$V(x) = x^T P x + a_1 x_1^4 + a_2 x_2^4. \tag{24}$$

The input is perturbed by producing uniform random variables k_1 and k_2 such that $1 < k_1 < 5$ and $0 < k_2 < 4$. Two sample CLFs can be written,

$$P^{(1)} = \begin{bmatrix} 1.055 & 0.222 \\ 0.222 & 0.261 \end{bmatrix}, a_1^{(1)} = 0.652, a_2^{(1)} = 0.124 \tag{25}$$

$$P^{(2)} = \begin{bmatrix} 0.641 & 0.201 \\ 0.201 & 0.599 \end{bmatrix}, a_1^{(2)} = 0.310, a_2^{(2)} = 0.386. \tag{26}$$

These two CLFs vary in the range of stabilizing inputs. This can be visualized for a particular state when $x = [-0.2, -0.2]^T$ as in Fig. 3. The two sets of stabilizable states $\mathcal{X}_{ps}^{(1)}$ and $\mathcal{X}_{ps}^{(2)}$ provided by Theorem 1 were approximated using a grid-based search and shown in Fig. 4.

The closed loop system was simulated with switching control from initial condition $x_0 = [-0.5, -0.5]^T$ with results in Fig. 5. CLF V_1 was active initially, and the CLF switched according to the third plot. The system is stabilized using the algorithm presented here despite frequent switching.

VII. Conclusions

This paper covered the problem of stabilizing a nonlinear system with input constraints using switching control. First, a result was presented that guaranteed the existence of a stabilizing set of inputs within the constraints assuming that a CLF is known. Then, a switching algorithm was presented that guarantees stability and is maximally permissive for the

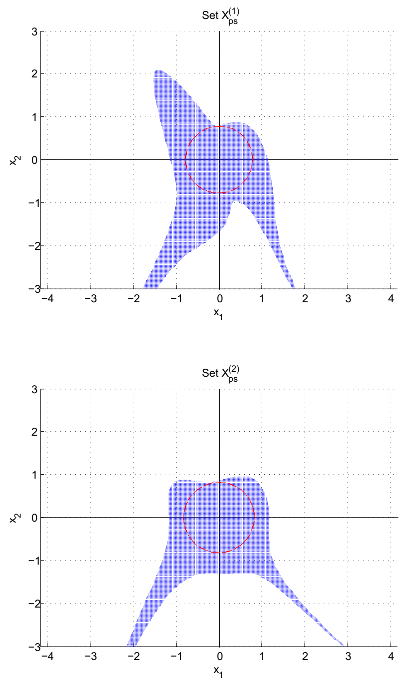

Fig. 4. The sets of stabilizable states $\mathcal{X}_{ps}^{(1)}$ and $\mathcal{X}_{ps}^{(2)}$ are shown for the CLFs V_1 and V_2, respectively.

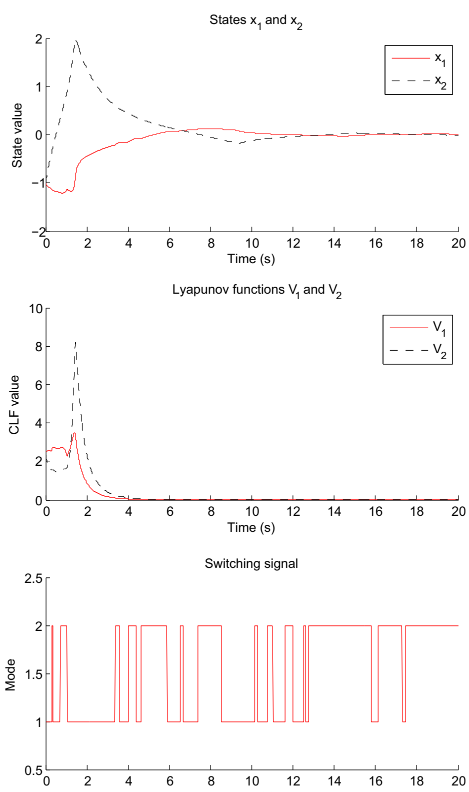

Fig. 5. The results of the simulation can be seen in three parts: the trajectories of states x_1 and x_2, the CLFs V_1 and V_2, and the switching signal $\sigma(t)$.

known CLFs for a system. These CLFs may be found using the LMI or SOS methods discussed in this paper. Finally, an example was provided to demonstrate the application of these methods to a sample system.

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